Existence and Stability of Periodic Solutions of a Lotka-Volterra System

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Abstract: In this paper, we study a Lotka-Volterra model which contains two prey and one predator with the Beddington-DeAngelis functional responses. First, we establish a set of sufficient conditions for existence of positive periodic solutions. Second, we investigate global asymptotic stability of boundary periodic solutions. Finally, we present some numerical examples.

Keywords: Lotka-Volterra system; Periodic solution; Asymptotic stability; Lyapunov function.

1. INTRODUCTION

The dynamical relationship between predators and prey has been studied by several authors for a long time. In those researches, various forms of functional responses have been used. Here a functional response means the average number of prey killed per individual predator per unit of time. Some biologists have argued that in many situation, especially when predators have to search for food, functional responses should depend on both prey's and predator's densities, see [1, 7, 12, 13] and references therein.

Let us consider a population of three species, say a Lotka-Volterra model, with the following properties:

(i) one species is a predator of two competitive other species.

(ii) the predator consumes prey with the functional response given by Beddington [2] and DeAngelis et al. [6]

There are many models having the property (i) or (ii) with diffusion in a constant environment [3-5], [11], [15-17]. However, natural environments are usually periodic in time due to the periodicity of seasons. Therefore, the parameters in these models should be periodic in time. This paper devotes to studying such a Lotka-Volterra model which is performed by a nonlinear system of differential equations:

$$\begin{aligned} x_1' &= x_1[a_1(t) - b_{11}(t)x_1 - b_{12}(t)x_2] \\ &\quad - \tfrac{c_1(t)x_1x_3}{\alpha(t)+\beta(t)x_1+\gamma(t)x_3}, \\ x_2' &= x_2[a_2(t) - b_{21}(t)x_1 - b_{22}(t)x_2] \\ &\quad - \tfrac{c_2(t)x_2x_3}{\alpha(t)+\beta(t)x_2+\gamma(t)x_3}, \\ x_3' &= x_3\Big[- a_3(t) + \tfrac{d_1(t)x_1}{\alpha(t)+\beta(t)x_1+\gamma(t)x_3} \\ &\quad + \tfrac{d_2(t)x_2}{\alpha(t)+\beta(t)x_2+\gamma(t)x_3}\Big]. \end{aligned} \tag{1.1}$$

Here $x_i(t)$ represents the population density of species X_i at time $t\,(i \geqslant 1)$, X_3 is a predator species and X_1, X_2 are competitive prey species. At time t, $a_i(t)$ is the intrinsic growth rate of $X_i\,(i = 1, 2)$ and $a_3(t)$ is the death rate of X_3; $b_{ij}(t)$ measures the amount of competition between X_1 and $X_2\,(i \neq j, i, j \leqslant 2)$, and $b_{ii}(t)\,(i \leqslant 2)$ measures the inhibiting effect of environment on X_i. The predator consumes prey with functional responses:

$$\frac{c_1(t)x_1x_3}{\alpha(t)+\beta(t)x_1+\gamma(t)x_3} \text{ and } \frac{c_2(t)x_2x_3}{\alpha(t)+\beta(t)x_2+\gamma(t)x_3};$$

and contributes to its growth with amounts:

$$\frac{d_1(t)x_1}{\alpha(t)+\beta(t)x_1+\gamma(t)x_3} \text{ and } \frac{d_2(t)x_2}{\alpha(t)+\beta(t)x_2+\gamma(t)x_3}.$$

Furthermore, we assume that the parameters $a_i(t), b_{ij}(t), c_i(t), d_i(t), \alpha(t), \beta(t), \gamma(t) (1 \leqslant i, j \leqslant 3)$ are ω-periodic and continuous in t and bounded below by some positive constants.

In the next section, we present our main results. First, we use the continuation theorem in coincidence degree theory to show existence of positive periodic solutions of (1.1). Second, by using Lyapunov functions we verify global asymptotic stability of boundary periodic solutions. Finally, we give numerical examples.

2. MAIN RESULTS

For biological reasons we only consider (1.1) with nonnegative initial values, i.e. $x_1(0), x_2(0), x_3(0) \geqslant 0$. Let $g(t)$ be a function, for a brevity, instead of writing $g(t)$ we write g. If g is a bounded continuous function on $\mathbb{R}$, we denote

$$g^u = \sup_{t\in\mathbb{R}}\ g(t),\ g^l = \inf_{t\in\mathbb{R}} g(t),$$

and $\hat{g} = \frac{1}{\omega}\int_0^\omega g(t)dt$, if g is a periodic function with period ω.

Definition 1: A nonnegative solution $x^*(t)$ of (1.1) is called a global asymptotic stable solution if it attracts any other solution $x(t)$ of (1.1) in the sense that

$$\lim_{t\to\infty} \sum_{i=1}^3 |x_i(t) - x_i^*(t)| = 0.$$

2.1. Existence of positive periodic solutions

In this subsection, we shall study existence of periodic solutions of (1.1). It is not difficult to verify global existence and uniqueness of nonnegative solutions of (1.1). To show the existence of a positive periodic solution, we shall use the continuation theorem in coincidence degree theory which has been used for some mathematical models of Lotka-Volterra type [10, 16] and references therein.

The following are some concepts and results taking from [8].

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces. A linear mapping $L\colon \mathcal{D}(L) \subset \mathbb{X} \to \mathbb{Y}$ is called *Fredholm* if it satisfies two conditions:

(i) $\operatorname{Im} L$ is closed and has finite codimension;
(ii) Ker L has finite dimension.

The *index* of L is the integer $\dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L$. If L is Fredholm of index zero, there exist continuous projections $P\colon \mathbb{X} \to \mathbb{X}$ and $Q\colon \mathbb{Y} \to \mathbb{Y}$ such that $\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$, and an isomorphism $J\colon\ \operatorname{Im} Q \to \operatorname{Ker} L$. It follows that
$$L_p = L|_{\mathcal{D}(L)\cap \operatorname{Ker} P}\colon (I-P)\mathbb{X} \to \operatorname{Im} L$$
is invertible. We denote the inverse of that map by K_p. Let Ω be an open bounded subset of $\mathbb{X}$. A continuous mapping $N\colon \mathbb{X} \to \mathbb{Y}$ is said to be L*-compact* on $\bar{\Omega}$ if the following two conditions take place:

(i) the mapping $QN\colon \bar{\Omega} \to \mathbb{Y}$ is continuous and bounded;
(ii) $K_p(I-Q)N\colon \bar{\Omega} \to \mathbb{X}$ is compact, i.e. it is continuous and $K_p(I-Q)N(\bar{\Omega})$ is relatively compact.

To introduce the definition of the degree of N in Ω, for simplicity we assume that $\mathbb{X} = \mathbb{R}^N$. Suppose furthermore that N is smooth on $\bar{\Omega}$. Let $p \notin \partial\Omega$ be a regular value of N, i.e. the equation $N(x) = p$ on $\bar{\Omega}$ has only a finite number of solutions $x_1, \dots, x_n \in \Omega$ with nonsingular $DN(x_i)$ for each $i = 1, \dots, n$ where $DN(x_i)$ is the Jacobi matrix of N at x_i. Then the degree $\deg(N, \Omega, p)$ of N in Ω at p is defined by the formula

$$\deg(N, \Omega, p) = \sum_{i=1}^{n} \operatorname{sgn}\{\det DN(x_i)\}.$$

Lemma 2 (Continuation theorem [8]) Let L be a Fredholm mapping of index 0. Assume that $N\colon \bar{\Omega} \to \mathbb{Y}$ is L-compact on $\bar{\Omega}$ and satisfies conditions:

(a) for each $\lambda \in (0,1)$ every solution of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$,

(b) $QNx \neq \mathbf{0}$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$, and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, \mathbf{0}\} \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\mathcal{D}(L) \cap \bar{\Omega}$.

We now put

$$\begin{aligned}
&L_{i1} = \ln \frac{\hat{a}_i}{\hat{b}_{ii}}, H_{i1} = L_{i1} + 2\hat{a}_i\omega \quad (i = 1, 2),\\
&L_{12} = \ln\Big\{\frac{\hat{a}_1 - \hat{b}_{12}e^{H_{21}} - \widehat{(\frac{c_1}{\gamma})}}{\hat{b}_{11}}\Big\},\\
&H_{12} = L_{12} - 2\hat{a}_1\omega,\\
&L_{22} = \ln\Big\{\frac{\hat{a}_2 - \hat{b}_{21}e^{H_{11}} - \widehat{(\frac{c_2}{\gamma})}}{\hat{b}_{22}}\Big\},\\
&H_{22} = L_{22} - 2\hat{a}_2\omega,\\
&L_{31} = \ln\Big\{\frac{\hat{d}_1 e^{H_{11}} + \hat{d}_2 e^{H_{21}} - \hat{a}_3\alpha^l}{\hat{a}_3\gamma^l}\Big\},\\
&H_{31} = L_{31} + 2\hat{a}_3\omega,\\
&L_{32} = \ln[(\hat{d}_1 - \hat{a}_3\beta^u)e^{H_{12}}\\
&\qquad + (\hat{d}_2 - \hat{a}_3\beta^u)e^{H_{22}} - 2\hat{a}_3\alpha^u] - \ln(2\hat{a}_3\gamma^u),\\
&H_{32} = L_{32} - 2\hat{a}_3\omega.
\end{aligned}$$

The convention here is that $\ln x = -\infty$ if $x \leqslant 0$. Under the conditions

$$\begin{cases}
\hat{b}_{11}\hat{b}_{22} \neq \hat{b}_{12}\hat{b}_{21},\\
\hat{a}_1 - \hat{b}_{12}e^{H_{21}} - \widehat{(\frac{c_1}{\gamma})} > 0,\\
\hat{a}_2 - \hat{b}_{21}e^{H_{11}} - \widehat{(\frac{c_2}{\gamma})} > 0,\\
\hat{d}_1 e^{H_{11}} + \hat{d}_2 e^{H_{21}} - \hat{a}_3\alpha^l > 0,\\
(\hat{d}_1 - \hat{a}_3\beta^u)e^{H_{12}} + (\hat{d}_2 - \hat{a}_3\beta^u)e^{H_{22}} > 2\hat{a}_3\alpha^u,
\end{cases} \tag{2.1}$$

we shall verify existence of an ω-periodic solution of (1.1).

Theorem 3: Let (2.1) be satisfied. Then (1.1) has at least one positive ω-periodic solution.

Proof: By putting $x_i(t) = e^{u_i(t)} (i \geqslant 1)$, (1.1) becomes

$$\begin{cases}
u_1' = a_1 - b_{11}e^{u_1} - b_{12}e^{u_2} - \frac{c_1 e^{u_3}}{\alpha + \beta e^{u_1} + \gamma e^{u_3}},\\
u_2' = a_2 - b_{21}e^{u_1} - b_{22}e^{u_2} - \frac{c_2 e^{u_3}}{\alpha + \beta e^{u_2} + \gamma e^{u_3}},\\
u_3' = -a_3 + \frac{d_1 e^{u_1}}{\alpha + \beta e^{u_1} + \gamma e^{u_3}} + \frac{d_2 e^{u_2}}{\alpha + \beta e^{u_2} + \gamma e^{u_3}}.
\end{cases} \tag{2.2}$$

Let

$$\begin{aligned}
\mathbb{X} &= \mathbb{Y}\\
&= \{u = (u_1, u_2, u_3)^T \in C^1(\mathbb{R}, \mathbb{R}^3) \text{ such that}\\
&\qquad u_i(s) = u_i(s + \omega) \text{ for } s \in \mathbb{R} \text{ and } i \geqslant 1\},
\end{aligned}$$

with norm

$$||u|| = \sum_{i=1}^{3} \max_{s \in [0,\omega]} |u_i(s)|, \qquad u \in \mathbb{X}.$$

Then both $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. Let

$$N\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}(s)=\begin{bmatrix}N_1(s)\\N_2(s)\\N_3(s)\end{bmatrix}$$

$$=\begin{bmatrix}a_1-b_{11}e^{u_1}-b_{12}e^{u_2}-\frac{c_1e^{u_3}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}\\a_2-b_{21}e^{u_1}-b_{22}e^{u_2}-\frac{c_2e^{u_3}}{\alpha+\beta e^{u_2}+\gamma e^{u_3}}\\-a_3+\frac{d_1e^{u_1}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}+\frac{d_2e^{u_2}}{\alpha+\beta e^{u_2}+\gamma e^{u_3}}\end{bmatrix},$$

$$L\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}u_1'\\u_2'\\u_3'\end{bmatrix},$$

$$P\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=Q\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}\frac{1}{\omega}\int_0^\omega u_1(s)ds\\\frac{1}{\omega}\int_0^\omega u_2(s)ds\\\frac{1}{\omega}\int_0^\omega u_3(s)ds\end{bmatrix}.$$

Hence,

$$\operatorname{Ker}L=\mathbb{R}^3,\operatorname{Im}L=\{u\in\mathbb{Y}\,|\int_0^\omega u_i(s)ds=0,\,i\geqslant 1\},$$

and $\dim\operatorname{Ker}L=3=\operatorname{codim}\operatorname{Im}L$. Then, it is easy to obtain the following conclusions.

1. L is a Fredholm mapping of index zero, since $\operatorname{Im}L$ is closed in $\mathbb{Y}$.
2. P and Q are continuous projections such that $\operatorname{Im}P=\operatorname{Ker}L,\operatorname{Im}L=\operatorname{Ker}Q=\operatorname{Im}(I-Q)$.
3. The generalized inverse (to L) $K_P\colon\ \operatorname{Im}L\to\mathcal{D}(L)\cap\operatorname{Ker}P$ exists and is given by

$$K_P\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}(\nu)=\begin{bmatrix}\int_0^\nu u_1(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^\nu u_1(s)dsd\nu\\\int_0^\nu u_2(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^\nu u_2(s)dsd\nu\\\int_0^\nu u_3(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^\nu u_3(s)dsd\nu\end{bmatrix}.$$

4. QN and $K_P(I-Q)N$ are continuous.
5. N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega\subset\mathbb{X}$.

Let us now find an appropriate open, bounded subset Ω for application of the continuation theorem. Obviously, from (2.1), $-\infty<L_{i2}\leqslant L_{i1}<\infty\,(i\geqslant 1)$. Corresponding to the equation $Lu=\lambda Nu,\lambda\in(0,1)$, we have

$$\begin{aligned}u_1'&=\lambda\Big[a_1-b_{11}e^{u_1}-b_{12}e^{u_2}-\frac{c_1e^{u_3}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}\Big],\\u_2'&=\lambda\Big[a_2-b_{21}e^{u_1}-b_{22}e^{u_2}-\frac{c_2e^{u_3}}{\alpha+\beta e^{u_2}+\gamma e^{u_3}}\Big],\\u_3'&=\lambda\Big[-a_3+\frac{d_1e^{u_1}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}\\&\qquad+\frac{d_2e^{u_2}}{\alpha+\beta e^{u_2}+\gamma e^{u_3}}\Big].\end{aligned}\tag{2.3}$$

Suppose that $(u_1,u_2,u_3)^T\in\mathbb{X}$ is an arbitrary solution of (2.3). Integrating both the hand sides of (2.3) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}\hat{a}_1\omega&=\int_0^\omega\Big[b_{11}e^{u_1}+b_{12}e^{u_2}+\frac{c_1e^{u_3}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}\Big]dt,\\\hat{a}_2\omega&=\int_0^\omega\Big[b_{21}e^{u_1}+b_{22}e^{u_2}+\frac{c_2e^{u_3}}{\alpha+\beta e^{u_2}+\gamma e^{u_3}}\Big]dt,\\\hat{a}_3\omega&=\int_0^\omega\Big[\frac{d_1e^{u_1}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}+\frac{d_2e^{u_2}}{\alpha+\beta e^{u_2}+\gamma e^{u_3}}\Big]dt.\end{aligned}\tag{2.4}$$

Combining the first equations of (2.3) and (2.4), we observe that

$$\begin{aligned}&\int_0^\omega|u_1'|dt\\&\leqslant\lambda\Big[\int_0^\omega a_1dt+\int_0^\omega b_{11}e^{u_1}dt\\&\qquad+\int_0^\omega b_{12}e^{u_2}dt+\int_0^\omega\frac{c_1e^{u_3}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}\Big]dt\\&<2\hat{a}_1\omega.\end{aligned}$$

Similarly, we have $\int_0^\omega|u_2'(t)|dt<2\hat{a}_2\omega$, and

$$\begin{aligned}&\int_0^\omega|u_3'(t)|dt\leqslant\lambda\Big[\int_0^\omega a_3dt+\int_0^\omega\frac{d_1e^{u_1}}{\alpha+\beta e^{u_1}+\gamma e^{u_3}}dt\\&+\int_0^\omega\frac{d_2e^{u_2}}{\alpha+\beta e^{u_2}+\gamma e^{u_3}}\Big]dt<2\hat{a}_3\omega.\end{aligned}$$

Since $u\in\mathbb{X}$, there exist $\xi_i,\eta_i\in[0,\omega]\,(i\geqslant 1)$ such that

$$u_i(\xi_i)=\min_{t\in[0,\omega]}u_i(t),\quad u_i(\eta_i)=\max_{t\in[0,\omega]}u_i(t).\tag{2.5}$$

From the first equation of (2.4) and (2.5), we obtain

$$\begin{aligned}\hat{a}_1\omega&\geqslant\int_0^\omega b_{11}e^{u_1(\xi_1)}dt+\int_0^\omega b_{12}e^{u_2(\xi_2)}dt\\&=\hat{b}_{11}\omega e^{u_1(\xi_1)}+\hat{b}_{12}\omega e^{u_2(\xi_2)},\end{aligned}$$

which implies that $u_1(\xi_1)<L_{11}$. Hence, for all $t\geqslant 0$

$$u_1(t)\leqslant u_1(\xi_1)+\int_0^\omega|u_1'(t)|dt<L_{11}+2\hat{a}_1\omega=H_{11}.$$

Similarly, we have $u_2(t)<H_{21}$ for all $t\geqslant 0$.
On the other hand, from the first equation of (2.4) and (2.5),

$$\begin{aligned}&\hat{a}_1\omega\\&\leqslant\int_0^\omega b_{11}e^{u_1(\eta_1)}dt+\int_0^\omega b_{12}e^{u_2(\eta_2)}dt+\int_0^\omega\frac{c_1(t)}{\gamma(t)}dt\\&=\hat{b}_{11}\omega e^{u_1(\eta_1)}+\hat{b}_{12}\omega e^{u_2(\eta_2)}+\widehat{(\frac{c_1}{\gamma})}\omega\\&\leqslant\hat{b}_{11}\omega e^{u_1(\eta_1)}+\hat{b}_{12}\omega e^{H_{21}}+\widehat{(\frac{c_1}{\gamma})}\omega.\end{aligned}$$

Hence,

$$u_1(t)\geqslant u_1(\eta_1)-\int_0^\omega|u_1'(t)|dt\geqslant H_{12},\qquad\forall t\geqslant 0.$$

Similarly, $u_2(t) \geqslant H_{22}$ for all $t \geqslant 0$. Therefore, by putting $B_i = \max\{|H_{i1}|, |H_{i2}|\}$, we conclude that

$$\max_{t\in[0,\omega]} |u_i(t)| \leqslant B_i, \qquad i = 1, 2.$$

Let us give estimates for $u_3(t)$. It follows from the third equation of (2.4) and (2.5) that

$$\begin{aligned}\hat{a}_3\omega &\leqslant \int_0^\omega \Big[\frac{d_1(t)e^{H_{11}}}{\alpha^l + \gamma^l e^{u_3(\xi_3)}} + \frac{d_2(t)e^{H_{21}}}{\alpha^l + \gamma^l e^{u_3(\xi_3)}}\Big]dt \\ &= \frac{[\hat{d}_1 e^{H_{11}} + \hat{d}_2 e^{H_{21}}]\omega}{\alpha^l + \gamma^l e^{u_3(\xi_3)}}\end{aligned}$$

and

$$\begin{aligned}\hat{a}_3\omega \geqslant& \int_0^\omega \Big[\frac{d_1(t)e^{H_{12}}}{\alpha^u + \beta^u e^{H_{12}} + \gamma^u e^{u_3(\eta_3)}} \\ &+ \frac{d_2(t)e^{H_{22}}}{\alpha^u + \beta^u e^{H_{22}} + \gamma^u e^{u_3(\eta_3)}}\Big]dt \\ =&\frac{\hat{d}_1 e^{H_{12}}\omega}{\alpha^u + \beta^u e^{H_{12}} + \gamma^u e^{u_3(\eta_3)}} \\ &+ \frac{\hat{d}_2 e^{H_{22}}\omega}{\alpha^u + \beta^u e^{H_{22}} + \gamma^u e^{u_3(\eta_3)}} \\ \geqslant&\frac{[\hat{d}_1 e^{H_{12}} + \hat{d}_2 e^{H_{22}}]\omega}{2\alpha^u + \beta^u[e^{H_{12}} + e^{H_{22}}] + 2\gamma^u e^{u_3(\eta_3)}}.\end{aligned}$$

Hence, $u_3(\xi_3) \leqslant L_{31}$ and $u_3(\eta_3) \geqslant L_{32}$. We then observe that

$$u_3(t) \leqslant u_3(\xi_3) + \textstyle\int_0^\omega |u_3'(t)|dt \leqslant H_{31}$$

and

$$u_3(t) \geqslant u_3(\eta_3) - \int_0^\omega |u_3'(t)|dt \geqslant H_{32}.$$

Therefore, by putting $B_3 = \max\{|H_{31}|, |H_{32}|\}$, we get

$$\max_{t\in[0,\omega]} |u_3| \leqslant B_3.$$

By the above estimates, for any solution $u \in \mathbb{X}$ of (2.4) we have $||u|| \leqslant \sum_{i=1}^3 B_i$. Clearly, $B_i\ (i \geqslant 1)$ are independent of λ. Take $B = \sum_{i=1}^4 B_i$ where B_4 is taken sufficiently large such that $B_4 \geqslant \sum_{i=1}^3 \sum_{j=1}^2 |L_{ij}|$. Let $\Omega = \{u \in \mathbb{X} \,|\, ||u|| < B\}$, then Ω satisfies the condition (a) of Lemma 2.

Let us verify that the condition (b) of Lemma 2 is also satisfied. Consider the homotopy

$$H_\mu(u) = \mu QN(u) + (1-\mu)G(u), \qquad \mu \in [0,1]$$

where $G\colon \mathbb{R}^3 \to \mathbb{R}^3$,

$$G(u) = \begin{bmatrix} \hat{a}_1 - \hat{b}_{11}e^{u_1} - \hat{b}_{12}e^{u_2} \\ \hat{a}_2 - \hat{b}_{21}e^{u_1} - \hat{b}_{22}e^{u_2} \\ -\hat{a}_3 + f(u_1,u_3) + g(u_2,u_3)\end{bmatrix}$$

with $f(u_1,u_3) = \frac{1}{\omega}\int_0^\omega \frac{d_1 e^{u_3}\,dt}{\alpha + \beta e^{u_1} + \gamma e^{u_3}}$ and $g(u_2,u_3) = \frac{1}{\omega}\int_0^\omega \frac{d_2 e^{u_3}\,dt}{\alpha + \beta e^{u_2} + \gamma e^{u_3}}$. It is easy to see that

$$H_\mu(u) = \begin{bmatrix} \hat{a}_1 - \hat{b}_{11}e^{u_1} - \hat{b}_{12}e^{u_2} - \frac{1}{\omega}\int_0^\omega \frac{\mu c_1 e^{u_3}\,dt}{\alpha + \beta e^{u_1} + \gamma e^{u_3}} \\ \hat{a}_2 - \hat{b}_{21}e^{u_1} - \hat{b}_{22}e^{u_2} - \frac{1}{\omega}\int_0^\omega \frac{\mu c_2 e^{u_3}\,dt}{\alpha + \beta e^{u_2} + \gamma e^{u_3}} \\ -\hat{a}_3 + f(u_1,u_3) + g(u_2,u_3)\end{bmatrix}.$$

By carrying out similar arguments as above, we observe that any solution u^* of the equation $H_\mu(u) = \mathbf{0} \in \mathbb{R}^3$ with $\mu \in [0,1]$ satisfies the estimate

$$L_{i2} \leqslant u_i^* \leqslant L_{i1}, \qquad i \geqslant 1. \tag{2.6}$$

Thus, $\mathbf{0} \notin H_\mu(\partial\Omega \cap \operatorname{Ker} L)$ for $\mu \in [0,1]$. Consequently, by taking $\mu = 1$, we conclude that $\mathbf{0} \notin QN(\partial\Omega \cap \operatorname{Ker} L)$. Note that the isomorphism J can be the identity mapping I, since $\operatorname{Im} P = \operatorname{Ker} L$. By the invariance property of homotopy, we obtain that

$$\begin{aligned}&\deg(JQN, \Omega \cap \operatorname{Ker} L, \mathbf{0}) = \deg(QN, \Omega \cap \operatorname{Ker} L, \mathbf{0}) \\ &= \deg(QN, \Omega \cap \mathbb{R}^3, \mathbf{0}) = \deg(G, \Omega \cap \mathbb{R}^3, \mathbf{0}) \\ &= \operatorname{sgn}\det\Lambda \\ &= \operatorname{sgn}\left\{(\hat{b}_{11}\hat{b}_{22} - \hat{b}_{12}\hat{b}_{21})\frac{\partial[f(u_1,u_3) + g(u_2,u_3)]}{\partial u_3}\right\}\end{aligned}$$

where

$$\Lambda = \begin{bmatrix} -\hat{b}_{11}e^{u_1} & -\hat{b}_{12}e^{u_2} & 0 \\ -\hat{b}_{21}e^{u_1} & -\hat{b}_{22}e^{u_2} & 0 \\ \frac{\partial f(u_1,u_3)}{\partial u_3} & \frac{\partial g(u_2,u_3)}{\partial u_3} & \frac{\partial f(u_1,u_3)}{\partial u_3} + \frac{\partial g(u_2,u_3)}{\partial u_3}\end{bmatrix}.$$

Since both functions $f(u_1,u_3)$ and $g(u_2,u_3)$ increase in u_3, $\frac{\partial f(u_1,u_3)}{\partial u_3} + \frac{\partial g(u_2,u_3)}{\partial u_3} > 0$. Hence, by using the first condition in (2.1), we conclude that

$$\deg(JQN, \Omega \cap \operatorname{Ker} L, \mathbf{0}) \neq 0.$$

By now we have proved that Ω verifies all requirements of Lemma 2. Therefore, the equation $Lu = Nu$ has at least one solution in $\mathcal{D}(L) \cap \bar{\Omega}$, i.e. (2.2) has at least one ω-periodic solution u^* in $\mathcal{D}(L) \cap \bar{\Omega}$. Set $x_i^* = e^{u_i^*} (i \geqslant 1)$, then x^* is an ω-periodic solution of (1.1) with strictly positive components. It completes the proof. ■

2.2. Global asymptotic stability of boundary periodic solutions

In this subsection, we shall establish a sufficient criteria for global asymptotic stability of boundary ω-periodic solutions of (1.1). Consider the boundary dynamics of (1.1) where X_3 is absent, i.e. $x_3(t) = 0$ for every $t \geqslant 0$. We then consider the periodic competitive model of two prey X_1, X_2:

$$\begin{cases} x_1' = x_1\left[a_1(t) - b_{11}(t)x_1 - b_{12}(t)x_2\right], \\ x_2' = x_2\left[a_2(t) - b_{21}(t)x_1 - b_{22}(t)x_2\right]. \end{cases} \tag{2.7}$$

Denote by $\bar{X}_i(t)$ the unique positive ω-periodic solution of the logistic equation:

$$X' = X\left[a_i(t) - b_{ii}(t)X\right].$$

Then $\bar{X}_i(t) = \frac{e^{\int_0^\omega a_i(s)ds - 1}}{\int_t^{t+\omega} b_{ii}(s)e^{-\int_s^t a(\tau)d\tau}ds}$. Due to [9], if

$$\hat{a}_i > \widehat{b_{ij}\bar{X}_j} \qquad (i \neq j, i,j = 1,2), \tag{2.8}$$

then (2.7) has a positive ω-periodic solution $(\bar{x}_1, \bar{x}_2)$. Furthermore, if

$$\hat{A}_{12} < 0 \tag{2.9}$$

then $(\bar{x}_1, \bar{x}_2)$ is globally asymptotically stable, where $a_{ij}(t) = b_{ij}(t)\bar{x}_j(t)$ $(i \neq j, i, j = 1, 2)$ and

$$A_{12}(t) = \max\left\{\frac{(a_{ij} + a_{ji})^2}{4a_{ii}} - a_{jj}, i \neq j\right\}.$$

Our result is as follows.

Theorem 4: If (2.8) and (2.9) hold then $\bar{x} = (\bar{x}_1, \bar{x}_2, 0)$ is a ω-periodic boundary solution of (1.1). Furthermore,

(i) If $b_{ij} < b_{jj}$ $(1 \leqslant i \neq j \leqslant 2)$ and $c_1 + c_2 + d_1 + d_2 < \beta a_3$ then $\bar{x}$ is globally asymptotically stable.

(ii) If $c_1 + c_2 + d_1 + d_2 < \beta a_3$ then $\bar{x}$ attracts any solution x of (1.1) which satisfies the condition

$$[x_1(t) - \bar{x}_1(t)][x_2(t) - \bar{x}_2(t)] \geqslant 0, \qquad \forall t \geqslant 0.$$

(iii) If $d_1 + d_2 < \beta a_3$ then $\bar{x}$ attracts any solution x of (1.1) which satisfies the condition

$$x_i(t) \geqslant \bar{x}_i(t), \qquad \forall t \geqslant 0, i = 1, 2.$$

Proof: The first statement is obvious. To prove (i), let x be any other solution of (1.1). Consider a Lyapunov function defined by $V(t) = \sum_{i=1}^{2} |\ln x_i - \ln \bar{x}_i| + x_3, t \geqslant 0$. Calculating of the right derivative $D^+V(t)$ of $V(t)$ along the solutions of (1.1) gives

$$\begin{aligned}
&D^+V(t)\\
&= \sum_{i=1}^{2} \operatorname{sgn}(x_i - \bar{x}_i)\left(\frac{x_i'}{x_i} - \frac{\bar{x}_i'}{\bar{x}_i}\right) + x_3'\\
&= \sum_{i \neq j}^{2} \Big\{ \operatorname{sgn}(x_i - \bar{x}_i)\Big[(a_i - b_{ii}x_i - b_{ij}x_j\\
&\quad - \frac{c_i x_i x_3}{\alpha + \beta x_i + \gamma x_3}) - (a_i - b_{ii}\bar{x}_i - b_{ij}\bar{x}_j)\Big]\Big\}\\
&\quad + (-a_3 + \sum_{i=1}^{2} \frac{d_i x_i}{\alpha + \beta x_i + \gamma x_3})x_3\\
&= \sum_{i \neq j}^{2} \{[-b_{ii}|x_i - \bar{x}_i| - b_{ij}(x_j - \bar{x}_j)\operatorname{sgn}(x_i - \bar{x}_i)]\}\\
&\quad + x_3\Big[- a_3 + \sum_{i=1}^{2} \Big\{ \frac{d_i x_i}{\alpha + \beta x_i + \gamma x_3}\\
&\qquad - \frac{c_i x_i \operatorname{sgn}(x_i - \bar{x}_i)}{\alpha + \beta x_i + \gamma x_3}\Big\}\Big]. \qquad (2.10)
\end{aligned}$$

Then

$$\begin{aligned}
D^+V(t) \leqslant &\sum_{i \neq j}^{2} (b_{ij} - b_{jj})|x_j - \bar{x}_j|\\
&+ \frac{(c_1 + c_2 + d_1 + d_2 - \beta a_3)x_3}{\beta}. \qquad (2.11)
\end{aligned}$$

By assumptions in (i) and the periodicity of parameters, there exist $\mu_1 > 0$ such that

$$\max_{t \in [0,\omega], 1 \leqslant i \neq j \leqslant 2} \left\{\frac{c_1 + c_2 + d_1 + d_2 - \beta a_3}{\beta}, b_{ij} - b_{jj}\right\} < -\mu_1.$$

Thus, by integrating both the hand sides of (2.11) from 0 to t, we observe that

$$V(t) + \mu_1 \int_0^t \sum_{i=1}^{3} |x_i - \bar{x}_i| ds \leqslant V(0) < \infty$$

for every $t \geqslant 0$. Hence, $\sum_{i=1}^{3} |x_i - \bar{x}_i| \in L^1([0, \infty))$.

On the other hand, by the periodicity, x_i and $\bar{x}_i$ $(i \geqslant 1)$ have bounded derivatives on $[0, \infty)$. As a consequence, $\sum_{i=1}^{3} |x_i - \bar{x}_i|$ is uniformly continuous on $[0, \infty)$. Therefore, by using the Barbalat lemma [16], we conclude that

$$\lim_{t \to \infty} \sum_{i=1}^{3} |x_i - \bar{x}_i| = 0,$$

i.e. $\bar{x}$ is globally asymptotically stable.

Similarly, we obtain the conclusions in (ii) and (iii) by using the following inequalities, respectively.

$$\begin{aligned}
&D^+V(t)\\
&= \sum_{i \neq j}^{2} [-(b_{ii} + b_{ji})|x_i - \bar{x}_i|] + x_3\Big[- a_3\\
&\quad + \sum_{i=1}^{2} \Big\{ \frac{d_i x_i}{\alpha + \beta x_i + \gamma x_3} - \frac{c_i x_i \operatorname{sgn}(x_i - \bar{x}_i)}{\alpha + \beta x_i + \gamma x_3}\Big\}\Big]\\
&\leqslant \sum_{i \neq j}^{2} [-(b_{ii} + b_{ji})|x_i - \bar{x}_i|]\\
&\quad + \frac{(c_1 + c_2 + d_1 + d_2 - \beta a_3)x_3}{\beta},
\end{aligned}$$

and

$$\begin{aligned}
&D^+V(t)\\
&= \sum_{i \neq j}^{2} [-(b_{ii} + b_{ji})|x_i - \bar{x}_i|] + x_3\Big[- a_3\\
&\quad + \sum_{i=1}^{2} \Big\{ \frac{d_i x_i}{\alpha + \beta x_i + \gamma x_3} - \frac{c_i x_i}{\alpha + \beta x_i + \gamma x_3}\Big\}\Big]\\
&\leqslant \sum_{i \neq j}^{2} [-(b_{ii} + b_{ji})|x_i - \bar{x}_i|] + \frac{(d_1 + d_2 - \beta a_3)x_3}{\beta}.
\end{aligned}$$

We complete the proof. ■

2.3. Numerical examples

In this subsection, we exhibit some numerical examples which show the convergence of positive solutions of (1.1) to periodic solutions of (1.1). Set $a_1 = 3 + \sin(8t); a_2 = 5.5 - 0.2\cos(8t); a_3 = 0.4 - 0.3\cos(8t); b_{11} = 2 + \cos(8t); b_{22} = 5 + 0.4\sin(8t); b_{12} = 0.04 - 0.02\sin(8t); b_{21} = 0.15 - 0.1\cos(8t); c_1 = 0.5 - 0.4\sin(8t);\ c_2 = 0.4 - 0.3\sin(8t); \alpha = 0.03 - 0.02\cos(8t); \beta = 0.3 + 0.2\cos(8t); \gamma = 2 - \sin(8t); d_1 = 3 + 2\sin(8t); d_2 = 3 - 2\sin(8t)$; and an initial value $(x_1(0), x_2(0), x_3(0)) = (0.5, 0.7, 1)$. Figure 1 shows the behavior of the solution of (1.1). It is seen that the solution converges to a positive periodic solution of (1.1).

We now set $a_3 = 4 - 0.3\cos(8t)$ and $\beta = 3 + 0.2\cos(8t)$ and retain other parameters as above. Figure 2 gives the behavior of the positive solution of (1.1). It converges to the boundary periodic solution of (1.1).

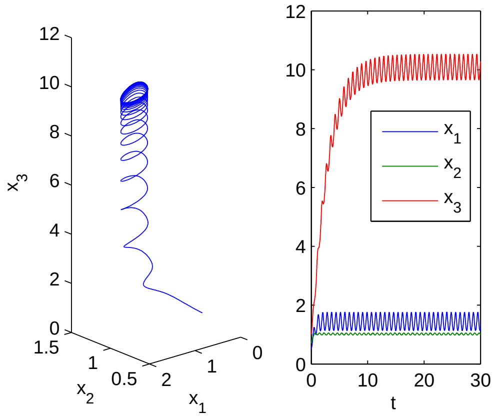

Fig. 1 A solution of (1.1) which converges to a positive periodic solution of (1.1)

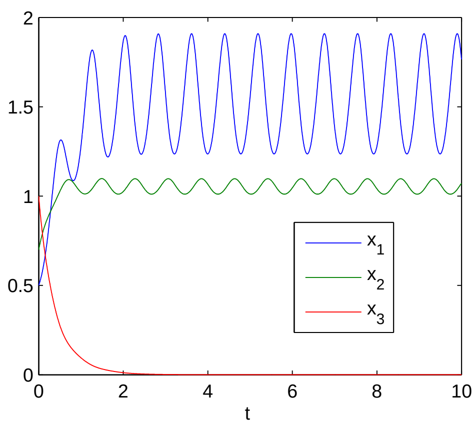

Fig. 2 A positive solution of (1.1) which converges to the boundary periodic solution of (1.1)

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